Chapter 6

Noise Specifications

6.1 Signal-to-Noise Ratio

The signal-to-noise ratio is usually measured at the output of an amplifier where the signal and noise voltages are larger and easier to measure. It is given by $SNR = v_{so}^2/v_{no}^2$, where v_{so}^2 is the mean-square signal output voltage and v_{no}^2 is the mean-square noise output voltage. It is usually specified in dB by the relation $10\log\left(v_{so}^2/v_{no}^2\right)$. When calculating the SNR with the $V_n - I_n$ amplifier noise model, it is convenient to make the calculation at the amplifier input. When the source is modeled by a Thévenin source, the SNR is given by $SNR = v_s^2/v_{ni}^2$, where v_s^2 is the mean-square source voltage and v_{ni}^2 is the mean-square equivalent input noise voltage. When it is modeled by a Norton source, it is given by $SNR = i_s^2/i_{ni}^2$, where i_s^2 is the mean-square source current and i_{ni}^2 is the mean-square equivalent input noise current. Expressions are derived below for the SNR for both cases. The source impedance and admittance which maximizes the SNR are also derived.

6.1.1 Thévenin Source

When the source is modeled by a Thévenin equivalent circuit as in Fig. 6.1, the signal-to-noise ratio is given by $SNR = v_s^2/v_{ni}^2$. When Eq. (4.3) is used for v_{ni}^2, it follows that the SNR is given by

$$SNR = \frac{v_s^2}{v_{ni}^2} = \frac{v_s^2}{4kTR_s\Delta f + v_n^2 + 2v_n i_n \operatorname{Re}\left(\gamma Z_s^*\right) + i_n^2\left|Z_s\right|^2} \tag{6.1}$$

where $Z_s = R_s + jX_s$ and $\gamma = \gamma_r + j\gamma_i$. It is expressed in dB by the relation $10\log\left(v_s^2/v_{ni}^2\right)$. It is maximized by minimizing v_{ni}^2. The source impedance which minimizes v_{ni}^2 can be obtained by setting $\partial v_{ni}^2/\partial R_s = 0$ and $\partial v_{ni}^2/\partial X_s = 0$ and solving for R_s and X_s. The solution for R_s is negative. Because this is not realizable, $R_s = 0$ is the realizable solution for the least noise. The source impedance which minimizes v_{ni}^2 is given by

$$Z_s = R_s + jX_s = 0 - j\gamma_i\frac{v_n}{i_n} \tag{6.2}$$

Because minimum noise occurs for $R_s = 0$, it can be concluded that a resistor should never be connected in series with a source at an amplifier input if noise performance is a design criterion. If a series resistor is required, e.g. for stability, it should be much smaller than R_s. Although the

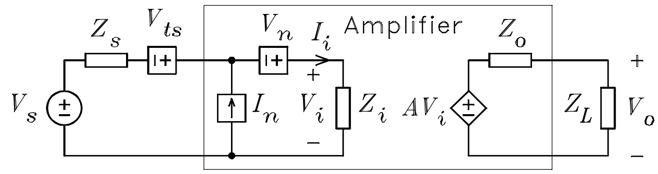

Figure 6.1: Amplifier with Thévenin source.

output impedance of a source is usually fixed, the SNR can be improved by adding a reactance in series with the source which makes the total series reactance equal to the imaginary part of Z_s in Eq. (6.2). When this is the case, v_{ni}^2 is given by

$$v_{ni}^2 = 4kTR_s\Delta f + v_n^2\left(1 - \gamma_i^2\right) + 2\gamma_r R_s v_n i_n + i_n^2 R_s^2 \tag{6.3}$$

6.1.2 Norton Source

When the source is modeled by a Norton equivalent circuit as in Fig. 6.2, the signal-to-noise ratio is given by $SNR = i_s^2/i_{ni}^2$. When Eq. (4.22) is used for i_{ni}^2, it follows that the SNR is given by

$$SNR = \frac{i_s^2}{i_{ni}^2} = \frac{i_s^2}{4kTG_s\Delta f + v_n^2\left|Y_s\right|^2 + 2v_n i_n \operatorname{Re}\left(\gamma Y_s\right) + i_n^2} \tag{6.4}$$

where $Y_s = G_s + jB_s$ and $\gamma = \gamma_r + j\gamma_i$. It is expressed in dB by the relation $10\log\left(i_s^2/i_{ni}^2\right)$. It is maximized by minimizing i_{ni}^2. The source admittance which minimizes i_{ni}^2 can be obtained by setting $\partial i_{ni}^2/\partial G_s = 0$ and $\partial i_{ni}^2/\partial B_s = 0$ and solving for G_s and B_s. The solution for G_s is negative. Because this is not realizable, $G_s = 0$ is the realizable solution for the least noise. The source admittance which minimizes i_{ni}^2 is given by

$$Y_s = G_s + jB_s = 0 + j\gamma_i \frac{i_n}{v_n} \tag{6.5}$$

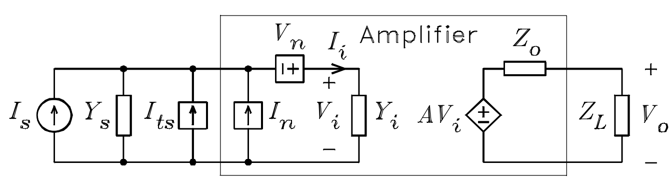

Figure 6.2: Amplifier with Norton source.

Because minimum noise occurs for $G_s = 0$, it can be concluded that a resistor should never be connected in parallel with a source at an amplifier input if noise performance is a design criterion. If a parallel resistor is required, e.g. as part of a bias network, it should be much larger than $1/G_s$. Although the output admittance of a source is usually fixed, the SNR can be improved by adding a susceptance in parallel with the source which makes the total parallel susceptance equal to the imaginary part of Y_s in Eq. (6.5). When this is the case, i_{ni}^2 is given by

$$i_{ni}^2 = 4kTG_s\Delta f + v_n^2 G_s^2 + 2\gamma_r G_s v_n i_n + i_n^2 \left(1 - \gamma_i^2\right) \tag{6.6}$$

6.2 Noise Factor and Noise Figure

The noise factor F of an amplifier is defined as the ratio of its actual SNR and the SNR if the amplifier is noiseless, where the temperature is taken to be the standard temperature T_0. When it is expressed in dB, it is called noise figure and is given by $NF = 10\log(F)$. In this section, the noise factor is derived for an amplifier driven by a Thévenin source and by a Norton source. Often, it is convenient to express F in terms of the amplifier noise resistance R_n and noise conductance G_n defined in Eqs. (2.18) and (2.19) and the correlation impedance Z_γ and correlation admittance Y_γ defined in Eqs. (3.12) and (3.13). These are related to v_n^2, i_n^2, and γ by

$$R_n = \frac{v_n^2}{4kT_0\Delta f} \tag{6.7}$$

$$G_n = \frac{i_n^2}{4kT_0\Delta f} \tag{6.8}$$

$$Z_\gamma = R_\gamma + jX_\gamma = \gamma\frac{v_n}{i_n} = (\gamma_r + j\gamma_i)\frac{v_n}{i_n} \tag{6.9}$$

$$Y_\gamma = G_\gamma + jB_\gamma = \gamma^*\frac{i_n}{v_n} = (\gamma_r - j\gamma_i)\frac{i_n}{v_n} \tag{6.10}$$

Note that R_n and G_n, respectively, represent normalized values of v_n^2 and i_n^2, where the normalization factor is $4kT_0\Delta f$.

6.2.1 Thévenin Source

Consider the amplifier model in Fig. 6.1. If the amplifier is noiseless, the signal-to-noise ratio given by $SNR = v_s^2/v_{ts}^2$, where v_s^2 is the mean-square source voltage and v_{ts}^2 is the mean-square thermal noise voltage generated by the source impedance. The noise factor F is obtained by dividing the noiseless amplifier SNR by Eq. (6.1) to obtain

$$F = \frac{\left(v_s^2/v_{ts}^2\right)}{\left(v_s^2/v_{ni}^2\right)} = \frac{v_{ni}^2}{v_{ts}^2} = 1 + \frac{v_n^2 + 2v_n i_n \operatorname{Re}\left(\gamma Z_s^*\right) + i_n^2\left|Z_s\right|^2}{4kT_0R_s\Delta f} \tag{6.11}$$

It follows from this expression that a noiseless amplifier has the noise factor $F = 1$. An alternate expression for F is obtained when the amplifier noise parameters are expressed in terms of R_n, G_n, and Z_γ. It is

$$F = 1 + \frac{R_n + 2G_n\operatorname{Re}\left(Z_\gamma Z_s^*\right) + G_n\left|Z_s\right|^2}{R_s} \tag{6.12}$$

The value of Z_s which minimizes F is called the optimum source impedance and is denoted by Z_{opt}. It is obtained by setting $\partial F/\partial R_s = 0$ and $\partial F/\partial X_s = 0$ and solving for R_s and X_s. The impedance is given by

$$Z_{opt} = R_{opt} + jX_{opt} = \left[\sqrt{1-\gamma_i^2} - j\gamma_i\right]\frac{v_n}{i_n} = \sqrt{\frac{R_n}{G_n} - X_\gamma^2} - jX_\gamma \tag{6.13}$$

Note that the imaginary part of Z_{opt} is equal to the imaginary part of Z_s in Eq. (6.2) which maximizes the signal-to-noise ratio. The corresponding minimum value of the noise factor is called the optimum noise factor and is given by

$$F_{min} = 1 + \frac{v_n i_n}{2kT_0 \Delta f}\left(\gamma_r + \sqrt{1-\gamma_i^2}\right) = 1 + 2G_n\left(R_\gamma + R_{opt}\right) \tag{6.14}$$

We next wish to express F in terms of F_{min} and Z_{opt}. It follows from Eqs. (6.12) and (6.14) that the difference $F - F_{min}$ is given by

$$\begin{aligned} F - F_{min} &= \frac{R_n + 2G_n\left(R_\gamma R_s + X_\gamma X_s\right) + G_n\left|Z_s\right|^2}{R_s} - 2G_n\left(R_\gamma + R_{opt}\right) \\ &= \frac{R_n - 2G_n\left(R_{opt} R_s + X_{opt} X_s\right) + G_n\left|Z_s\right|^2}{R_s} \end{aligned} \tag{6.15}$$

where $X_\gamma = -X_{opt}$ has been used. The square in the numerator of this expression can be completed by adding and subtracting the term $G_n\left(R_{opt}^2 + X_{opt}^2\right) = G_n\left|Z_{opt}\right|^2$. This leads to the equation

$$F - F_{min} = \frac{R_n + G_n\left[\left(R_s - R_{opt}\right)^2 + \left(X_s - X_{opt}\right)^2 - \left|Z_{opt}\right|^2\right]}{R_s} \tag{6.16}$$

From Eq. (6.13), we have $\left|Z_{opt}\right|^2 = R_n/G_n$. It follows that F can be written

$$\begin{aligned} F &= F_{min} + \frac{G_n}{R_s}\left[\left(R_s - R_{opt}\right)^2 + \left(X_s - X_{opt}\right)^2\right] \\ &= F_{min} + \frac{G_n}{R_s}\left|Z_s - Z_{opt}\right|^2 \end{aligned} \tag{6.17}$$

If G_n, F_{min}, and $Z_{opt} = R_{opt} + jX_{opt}$ are given for an amplifier, it can be shown that

$$i_n^2 = 4kTG_n\Delta f \tag{6.18}$$

$$\gamma_i = \frac{-\operatorname{sgn}\left(X_{opt}\right)}{\sqrt{1 + \left(R_{opt}/X_{opt}\right)^2}} \tag{6.19}$$

$$v_n^2 = \left(\frac{X_{opt}}{\gamma_i}\right)^2 i_n^2 \tag{6.20}$$

$$\gamma_r = \frac{2kT_0\Delta f}{v_n i_n}\left(F_{min} - 1\right) - \sqrt{1-\gamma_i^2} \tag{6.21}$$

where $\operatorname{sgn}\left(X_{opt}\right) = X_{opt}/\left|X_{opt}\right|$.

The noise factor can also be expressed as a function of the reflection coefficients of the source and the amplifier input. Imagine a zero length transmission line having a characteristic impedance Z_c connecting the source to the amplifier. In Eq. (6.17), let

$$Z_s = Z_c\frac{1+\Gamma_s}{1-\Gamma_s} \tag{6.22}$$

$$Z_{opt} = Z_c\frac{1+\Gamma_{opt}}{1-\Gamma_{opt}} \tag{6.23}$$

$$R_s = \frac{1}{2}(Z_s + Z_s^*) = Z_c \frac{1 - |\Gamma_s|^2}{|1 - \Gamma_s|^2} \tag{6.24}$$

where the reflection coefficients Γ_s and Γ_{opt} are given by

$$\Gamma_s = \frac{Z_s - Z_c}{Z_s + Z_c} \tag{6.25}$$

$$\Gamma_{opt} = \frac{Z_{opt} - Z_c}{Z_{opt} + Z_c} \tag{6.26}$$

When the expressions for Z_s, Z_{opt}, and R_s are substituted into Eq. (6.17), the expression for F reduces to

$$F = F_{min} + \frac{4G_n Z_c |\Gamma_s - \Gamma_{opt}|^2}{|1 - \Gamma_{opt}|^2 \left(1 - |\Gamma_s|^2\right)} \tag{6.27}$$

Example 1 *Calculate F and NF for the amplifier in Example 1 in Chapter 4 for which $R_s = 75\,\Omega$ and $v_{ni} = 5.69\,\mathrm{nV}$. Assume $\Delta f = 1\,\mathrm{Hz}$ and $T = T_0 = 290\,\mathrm{K}$.*

Solution. The mean-square thermal noise voltage of the source is $v_{ts}^2 = 4kTR_s = 1.20 \times 10^{-18}\,\mathrm{V}^2$. Thus the noise factor and noise figure are

$$F = \frac{v_{ni}^2}{v_{ts}^2} = \frac{\left(5.69 \times 10^{-9}\right)^2}{1.2 \times 10^{-18}} = 27.0 \qquad NF = 10\log(27.0) = 14.3\,\mathrm{dB}$$

6.2.2 Norton Source

Consider the amplifier model in Fig. 6.2. If the amplifier is noiseless, the signal-to-noise ratio given by $SNR = i_s^2/i_{ts}^2$, where i_s^2 is the mean-square source current and i_{ts}^2 is the mean-square thermal noise current generated by the source impedance. The noise factor F is obtained by dividing the noiseless amplifier SNR by Eq. (6.4) to obtain

$$F = \frac{\left(i_s^2/i_{ts}^2\right)}{\left(i_s^2/i_{ni}^2\right)} = \frac{i_{ni}^2}{i_{ts}^2} = 1 + \frac{v_n^2 |Y_s|^2 + 2v_n i_n \operatorname{Re}(\gamma Y_s) + i_n^2}{4kT_0 G_s \Delta f} \tag{6.28}$$

An alternate expression for F is obtained when the amplifier noise parameters are expressed in terms of R_n, G_n, and Y_γ. It is

$$F = 1 + \frac{R_n |Y_s|^2 + 2R_n \operatorname{Re}(Y_\gamma Y_s) + G_n}{G_s} \tag{6.29}$$

The value of Y_s that minimizes F is called the optimum source admittance and is denoted by Y_{opt}. It is obtained by setting $\partial F/\partial G_s = 0$ and $\partial F/\partial B_s = 0$ and solving for G_s and B_s. The admittance is given by

$$Y_{opt} = G_{opt} + jB_{opt} = \left[\sqrt{1 - \gamma_i^2} + j\gamma_i\right]\frac{i_n}{v_n} = \sqrt{\frac{G_n}{R_n} - B_\gamma^2} + jB_\gamma \tag{6.30}$$

Note that this is the reciprocal of the optimum source impedance, i.e. $Y_{opt} = 1/Z_{opt}$. Also, the imaginary part of Y_{opt} is equal to the imaginary part of Y_s in Eq. (6.5) which maximizes the signal-to-noise ratio. The corresponding minimum value of the noise factor is given by

$$F_{min} = 1 + \frac{v_n i_n}{2kT_0\Delta f}\left(\gamma_r + \sqrt{1-\gamma_i^2}\right) = 1 + 2R_n\left(G_\gamma + G_{opt}\right) \tag{6.31}$$

We next wish to express F in terms of F_{min} and Y_0. It follows from Eqs. (6.29) and (6.31) that the difference $F - F_{min}$ is given by

$$\begin{aligned} F - F_{min} &= \frac{R_n\left|Y_s\right|^2 + 2R_n\left(G_\gamma G_s - B_\gamma B_s\right) + G_n}{G_s} - 2R_n\left(G_\gamma + G_{opt}\right) \\ &= \frac{R_n\left|Y_s\right|^2 - 2R_n\left(G_{opt} G_s + B_{opt} B_s\right) + G_n}{G_s} \end{aligned} \tag{6.32}$$

where $B_\gamma = B_{opt}$ has been used. The square in the numerator of this expression can be completed by adding and subtracting the term $R_n\left(G_{opt}^2 + B_{opt}^2\right) = R_n\left|Y_{opt}\right|^2$. This leads to the equation

$$F - F_{min} = \frac{R_n\left[\left(G_s - G_{opt}\right)^2 + \left(B_s - B_{opt}\right)^2 - \left|Y_0\right|^2\right] + G_n}{G_s} \tag{6.33}$$

From Eq. (6.30), we have $\left|Y_{opt}\right|^2 = G_n/R_n$. It follows that F can be written

$$\begin{aligned} F &= F_{min} + \frac{R_n}{G_s}\left[\left(G_s - G_{opt}\right)^2 + \left(B_s - B_{opt}\right)^2\right] \\ &= F_{min} + \frac{R_n}{G_s}\left|Y_s - Y_{opt}\right|^2 \end{aligned} \tag{6.34}$$

If R_n, F_{min}, and $Y_{opt} = G_{opt} + jB_{opt}$ are given for an amplifier, it can be shown that

$$v_n^2 = 4kTR_n\Delta f \tag{6.35}$$

$$\gamma_i = \frac{\operatorname{sgn}\left(B_{opt}\right)}{\sqrt{1 + \left(G_{opt}/B_{opt}\right)^2}} \tag{6.36}$$

$$i_n^2 = \left(\frac{B_{opt}}{\gamma_i}\right)^2 v_n^2 \tag{6.37}$$

$$\gamma_r = \frac{2kT_0\Delta f}{v_n i_n}\left(F_{min} - 1\right) - \sqrt{1-\gamma_i^2} \tag{6.38}$$

where $\operatorname{sgn}\left(B_{opt}\right) = B_{opt}/\left|B_{opt}\right|$.

The noise factor can also be expressed as a function of the reflection coefficients of the source and the amplifier input. Imagine a zero length transmission line having a characteristic admittance Y_c connecting the source to the amplifier. In Eq. (6.34), let

$$Y_s = Y_c\frac{1-\Gamma_s}{1+\Gamma_s} \tag{6.39}$$

$$Y_{opt} = Y_c\frac{1-\Gamma_{opt}}{1+\Gamma_{opt}} \tag{6.40}$$

$$G_s = \frac{1}{2}\left(Y_s + Y_s^*\right) = Y_c \frac{1 - |\Gamma_s|^2}{|1 + \Gamma_s|^2} \tag{6.41}$$

where the reflection coefficients Γ_s and Γ_{opt} are given by

$$\Gamma_s = \frac{Y_c - Y_s}{Y_c + Y_s} \tag{6.42}$$

$$\Gamma_{opt} = \frac{Y_c - Y_{opt}}{Y_c + Y_{opt}} \tag{6.43}$$

When the expressions for Y_s, Y_{opt}, and G_s are substituted into Eq. (6.34), the expression for F reduces to

$$F = F_{min} + \frac{4R_n Y_c \left|\Gamma_s - \Gamma_{opt}\right|^2}{\left|1 + \Gamma_{opt}\right|^2 \left(1 - |\Gamma_s|^2\right)} \tag{6.44}$$

6.2.3 The Noise Factor Fallacy

The noise factor can be a misleading specification. If an attempt is made to minimize F by adding resistors either in series or in parallel with the source at the input of an amplifier, the SNR is always decreased. This is referred to as the noise factor fallacy or the noise figure fallacy. Potential confusion can be avoided if low-noise amplifiers are designed to maximize the SNR. This is accomplished by minimizing the equivalent noise input voltage. If a series resistor must be included at the amplifier input, its value should be much smaller than the source impedance. If a parallel resistor must be included at the amplifier input, its value should be much larger than the source impedance. The following example illustrates the noise factor fallacy.

Example 2 *An amplifier has an input resistance* $R_i = 150\,\Omega$*. For* $\Delta f = 1\,\text{Hz}$*, its noise parameters are* $v_n = 2\,\text{nV}$*,* $i_n = 10\,\text{pA}$*, and* $\rho = 0.1$*. It is driven from a source having an output resistance* $R_s = 50\,\Omega$*. (a) Calculate* v_{ni}^2*,* F*, and* NF*. (b) A resistance is added in series with the source impedance to minimize* F*. Calculate the new* v_{ni}^2*,* F*, and* NF*. (c) Calculate the changes in the noise figure and the signal-to-noise ratio.*

Solution. (a)

$$v_{ni}^2 = 4kTR_s\Delta f + v_n^2 + 2\rho v_n i_n R_s + i_n^2 R_s^2 = 5.25 \times 10^{-18}\,\text{V}^2$$

$$F = \frac{v_{ni}^2}{4kTR_s\Delta f} = 6.56$$

$$NF = 10\log(F) = 8.17\,\text{dB}$$

(b)

$$R_{opt} = \frac{v_n}{i_n} = 200\,\Omega$$

$$v_{ni}^2 = 4kTR_{opt}\Delta f + v_n^2 + 2\rho v_n i_n R_{opt} + i_n^2 R_{opt}^2 = 1.2 \times 10^{-17}\,\text{V}^2$$

$$F = \frac{v_{ni}^2}{4kTR_{opt}\Delta f} = 3.75$$

$$NF = 10\log(F) = 5.74\,\text{dB}$$

(c) The decrease in the noise figure is $8.171 - 5.740 = 2.431\,\text{dB}$. The dB decrease in the SNR is $10\log\left(1.2 \times 10^{-17}/5.25 \times 10^{-18}\right) = 3.59\,\text{dB}$.

The above example illustrates how the noise figure appears to be decreased by adding resistance in series with an amplifier input. However, the signal-to-noise ratio is lowered. The fallacy comes from treating the added resistance as part of the source rather than part of the amplifier. In reality, the amplifier noise is increased by the added resistor, but the source noise remains constant. The correct way to calculate the noise factor with the added resistor is $F = v_{ni}^2/4kTR_s\Delta f$ which gives $F = 15$ and $NF = 11.76\,\text{dB}$. In this case, the noise figure decreases by $11.76 - 8.17 = 3.59\,\text{dB}$. This is the same as the dB decrease in the signal-to-noise ratio.

6.3 Noise Matching Network Examples

In the following examples, it is assumed that the source impedance is real. If it is not, a reactive element can be used in series or in parallel with the source to cancel the reactive component. Alternately, the reactive component in the matching network that is closest to the source can be adjusted to cancel the source reactance. The examples assume that the source is modeled by a Thévenin equivalent circuit. It is straightforward to apply the networks to the case where the source is modeled by a Norton equivalent circuit.

Example 3 *With a $50\,\Omega$ transmission line test fixture, the reflection coefficient seen looking out of the input terminals of an amplifier that minimizes its noise factor F at* 900 MHz *is determined to be $\Gamma_{opt} = 0.7\angle 23.7^\circ$. Determine the lengths ℓ_1 and ℓ_2 of the transmission lines in the network of Fig. 6.3 which cause the source impedance seen by the amplifier to be the optimum source impedance Z_{opt}. The source impedance is $R_s = 50\,\Omega$. The characteristic impedance of the two transmission lines in the network is $Z_{c1} = Z_{c2} = 75\,\Omega$.*

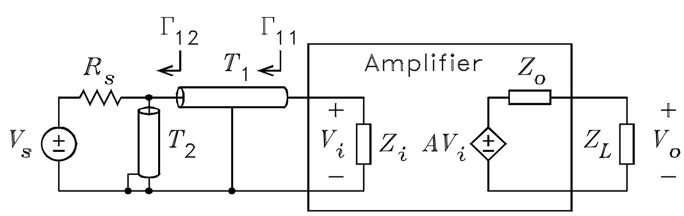

Figure 6.3: Transmission line noise matching network for Example 3.

Solution. With $Z_c = 50\,\Omega$, i.e. the line impedance of the text fixture used to measure Γ_{opt}, the optimum source impedance is given by

$$Z_{opt} = Z_c \frac{1 + \Gamma_{opt}}{1 - \Gamma_{opt}} \tag{6.45}$$

It follows that

$$Z_{opt} = 50 \frac{1 + (0.7\cos 23.7^\circ + j0.7\sin 23.7^\circ)}{1 - (0.7\cos 23.7^\circ - j0.7\sin 23.7^\circ)} = 122.6 + j135.2\,\Omega$$

The reflection coefficient at the amplifier end of T_1 is given by

$$\Gamma_{11} = \frac{Z_{opt} - Z_{c1}}{Z_{opt} + Z_{c1}} \tag{6.46}$$

which yields

$$\Gamma_{11} = \frac{(122.6 + j135.2) - 75}{(122.6 + j135.2) + 75} = 0.4230 + j0.3539$$

This has the magnitude

$$|\Gamma_{11}| = 0.5988$$

It follows from Eq. (5.156) that the reactance of T_2 is given by

$$X_2 = \pm Z_{c1} \left[\frac{1 - |\Gamma_{11}|^2}{|\Gamma_{11}|^2 (1 + Z_{c1}/R_s)^2 - (1 - Z_{c1}/R_s)^2} \right]^{1/2} \tag{6.47}$$

If the positive solution is used, this gives

$$X_2 = 75 \left[\frac{1 - 0.5988^2}{0.5988^2 (1 + 75/50)^2 - (1 - 75/50)^2} \right]^{1/2} = 42.58\,\Omega$$

By Eq. (5.157), the electrical length of T_2 is given by

$$\beta\ell_2 = \tan^{-1}\left(\frac{X_2}{Z_{c2}}\right) \tag{6.48}$$

Thus we have

$$\beta\ell_2 = \tan^{-1}\left(\frac{42.58}{75}\right) = 29.58^\circ$$

By Eq. (5.154), the reflection coefficient at the source end of T_1 is given by

$$\Gamma_{12} = \frac{(R_s \| jX_2) - Z_{c1}}{(R_s \| jX_2) + Z_{c1}} \tag{6.49}$$

which yields

$$\Gamma_{12} = \frac{(50 \| j42.58) - 75}{(50 \| j42.58) + 75} = -0.4654 + j0.3767$$

It follows from Eq. (5.160) that the electrical length of T_1 is given by

$$\beta\ell_1 = \frac{1}{2} \arg\left(\frac{\Gamma_{12}}{\Gamma_{11}}\right) \tag{6.50}$$

which gives

$$\beta\ell_1 = \frac{1}{2} \arg\left(\frac{-0.4654 + j0.3767}{0.4830 + j0.3539}\right) = 52.39^\circ$$

As a check of the impedance seen looking into T_1, we have

$$\begin{aligned} Z &= Z_{c1} \frac{(R_s \| jX_2) + jZ_{c1} \tan(\beta\ell_2)}{Z_{c1} + j (R_s \| jX_2) \tan(\beta\ell_2)} \\ &= 75 \frac{21.02 + j122.0}{42.96 + j27.28} = 122.6 + j135.2\,\Omega \end{aligned}$$

which is the desired value.

Example 4 *For the amplifier of Example 3, determine X_1 and X_2 in the noise matching network shown in Fig. 6.4.*

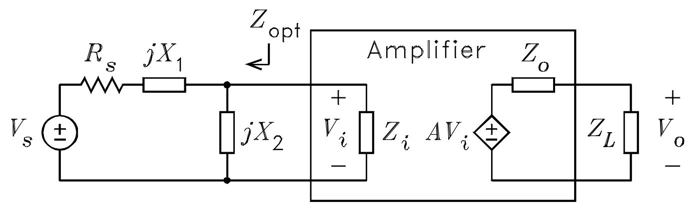

Figure 6.4: Lumped element noise matching network for Example 4.

Solution. The source impedance is $R_s = 50\,\Omega$. The optimum source impedance is $Z_{opt} = R_{opt} + jX_{opt} = 122.6 + j135.2\,\Omega$. To transform R_s into Z_{opt}, it can be shown that X_1 is given by

$$X_1 = \pm R_s \sqrt{\frac{R_{opt}}{R_s}\left[1 + \left(\frac{X_{opt}}{R_{opt}}\right)^2\right] - 1} \tag{6.51}$$

This equation gives $X_1 = \pm 105.3\,\Omega$. The solution for X_2 is

$$X_2 = \frac{R_s^2 + X_1^2}{(R_s X_{opt}/R_{opt}) - X_1} \tag{6.52}$$

For $X_1 = +105.3\,\Omega$, this gives $X_2 = -270.9\,\Omega$. For $X_1 = -105.3\,\Omega$, it gives $X_2 = +84.68\,\Omega$.

Example 5 *At a frequency of* 900 MHz, *it is determined that the optimum source impedance for an amplifier is $Z_{opt} = 30 - j60\,\Omega$. If $R_s = 50\,\Omega$, determine the values of X_1 and X_2 in the noise matching network showin in Fig. 6.5.*

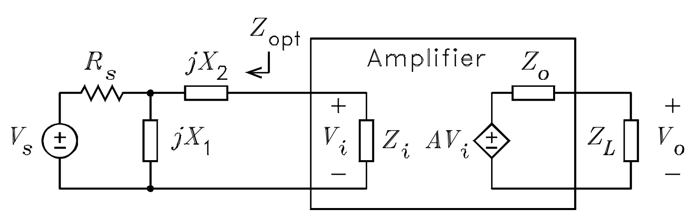

Figure 6.5: Lumped element noise matching network for Example 5.

Solution. To transform R_s into Z_{opt},it can be shown that X_1 is given by

$$X_1 = \frac{\pm R_s}{\sqrt{(R_s/R_{opt}) - 1}} \tag{6.53}$$

This equation gives $X_1 = \pm 61.2\,\Omega$. The reactance X_2 is given by

$$X_2 = X_{opt} - \left(1 - \frac{R_{opt}}{R_s}\right) X_1 \tag{6.54}$$

For $X_1 = +61.2\,\Omega$, this gives $X_2 = -84.5\,\Omega$. For $X_1 = -61.2\,\Omega$, it gives $X_2 = -35.5\,\Omega$.

6.4 Noise Temperature

The internal noise generated by an amplifier can be expressed as an equivalent input-termination noise temperature. When the source is represented by a Thévenin equivalent circuit, the noise temperature T_n is the temperature of the source resistance that generates a thermal noise voltage equal to the internal noise generated in the amplifier when referred to its input. For the Thévenin source, the noise temperature is defined by

$$4kT_nR_s\Delta f = v_n^2 + 2v_ni_n\,\mathrm{Re}\left(\gamma Z_s^*\right) + i_n^2\left|Z_s\right|^2 \tag{6.55}$$

where $R_s = \mathrm{Re}\left(Z_s\right)$. It follows that the noise temperature is given by

$$T_n = \frac{v_n^2 + 2v_ni_n\,\mathrm{Re}\left(\gamma Z_s^*\right) + i_n^2\left|Z_s\right|^2}{4kR_s\Delta f} \tag{6.56}$$

When the source is represented by a Norton equivalent circuit, the noise temperature is the temperature of the source conductance that generates a thermal noise current equal to the internal noise generated in the amplifier when referred to its input. It is given by

$$T_n = \frac{v_n^2\left|Y_s\right|^2 + 2v_ni_n\,\mathrm{Re}\left(\gamma Y_s\right) + i_n^2}{4kG_s\Delta f} \tag{6.57}$$

where $G_s = \mathrm{Re}\left(Y_s\right)$. The noise temperature is related to the noise factor by

$$T_n = \left(F - 1\right)T_0 \tag{6.58}$$

This holds for either the Thévenin or the Norton source.

Example 6 *Calculate the noise temperature of the amplifier of Example 2 for which $F = 6.56$.*

Solution. $T_n = (6.56 - 1) \times 290 = 1612\,\mathrm{K}$.

6.5 Noise Factor of a Multistage Amplifier

The circuit model for a multi-stage amplifier is shown in Fig. 6.6. It is shown in Section 4.3 that the equivalent noise input voltage is given by

$$V_{ni} = V_{ni1} + \frac{V_{ni2}}{G_{m1}Z_{o1}} + \cdots + \frac{V_{niN}}{G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}} \tag{6.59}$$

where $V_{ni1} = V_{ts} + V_{n1} + I_{n1}Z_s$, $V_{nij} = V_{nj} + I_{nj}Z_{o(j-1)}$ for $2 \le j \le N$, and N is the number of stages. The equivalent transconductance G_{mj} is the ratio of the short-circuit output current from the jth stage to its open-circuit input voltage. It is given by

$$G_{mj} = \frac{I_{oj}}{V_{ij(oc)}} = \frac{g_{mj}Z_{ij}}{Z_{o(j-1)} + Z_{ij}} \tag{6.60}$$

where $g_{mj} = I_{oj}/V_{ij}$ andV_{ij} is the input voltage across Z_{ij}. The noise factor of the amplifier is given by

$$\begin{aligned} F &= \frac{v_{ni}^2}{4kTR_s\Delta f} \\ &= \frac{1}{4kTR_s\Delta f}\left[v_{ni1}^2 + \frac{v_{ni2}^2}{\left|G_{m1}Z_{o1}\right|^2} + \cdots \right. \\ &\quad \left. + \frac{v_{niN}^2}{\left|G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}\right|^2}\right] \end{aligned} \tag{6.61}$$

where $R_s = \mathrm{Re}\left(Z_s\right)$.

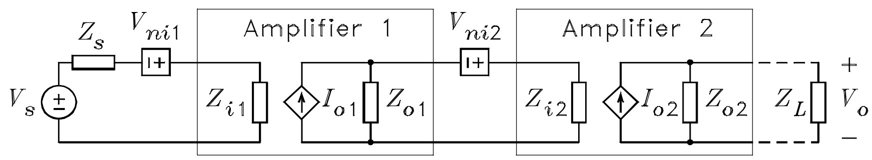

Figure 6.6: Multi-stage amplifier.

We wish to express F as a function of the noise factor of each stage. The noise factors are given by

$$\begin{aligned} F_1 &= \frac{v_{ni1}^2}{4kTR_s\Delta f} \\ F_2 &= \frac{4kTR_{o1}\Delta f + v_{ni2}^2}{4kTR_{o1}\Delta f} = 1 + \frac{v_{ni2}^2}{4kTR_{o1}\Delta f} \\ &\vdots \\ F_N &= \frac{4kTR_{o(N-1)}\Delta f + v_{niN}^2}{4kTR_{o1(N-1)}\Delta f} = 1 + \frac{v_{niN}^2}{4kTR_{o(N-1)}\Delta f} \end{aligned} \tag{6.62}$$

where $R_{oj} = \mathrm{Re}\left(Z_{oj}\right)$. Note that v_{ni1}^2 includes the thermal noise generated by R_s. For $j \geq 2$, v_{nij}^2 does not include the thermal noise of $R_{o(j-1)}$ because its noise is included in $v_{ni(j-1)}^2$. It follows from Eq. (6.61) and (6.62) that F can be written

$$\begin{aligned} F &= F_1 + \frac{\left(F_2 - 1\right)R_{o1}}{\left|G_{m1}Z_{o1}\right|^2 R_s} + \cdots \\ &\quad + \frac{\left(F_N - 1\right)R_{o(N-1)}}{\left|G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}\right|^2 R_s} \end{aligned} \tag{6.63}$$

We next express F as a function of the available power gains of the stages. For the first stage, the current through Z_{i1} is $I_{i1} = V_s/\left(Z_s + Z_{i1}\right)$. The power delivered by the source to Z_{i1} is given by

$$P_{i1} = i_{i1}^2 R_{i1} = \frac{v_s^2 R_{i1}}{\left|Z_s + Z_{i1}\right|^2} = \frac{v_s^2 R_{i1}}{\left(R_s + R_{i1}\right)^2 + \left(X_s + X_{i1}\right)^2} \tag{6.64}$$

The maximum value of P_{i1} is called the available input power P_{ai1}. It is solved for by setting $\partial P_{i1}/\partial R_{i1} = 0$ and $\partial P_{i1}/\partial X_{i1} = 0$ and solving for R_{i1} and X_{i1}. The solutions are $R_{i1} = R_s$ and $X_{i1} = -X_s$, i.e. $Z_{i1} = Z_s^*$. It follows that P_{ai1} is given by

$$P_{ai1} = \frac{v_s^2}{4R_s} \tag{6.65}$$

For the second stage, the current through Z_{i2} is $I_{i2} = I_{o1}Z_{o1}/(Z_{o1} + Z_{i2})$, where $I_{o1} = G_{m1}V_s$. The power delivered to Z_{i2} is the power output from the first stage. It is given by

$$\begin{aligned} P_{o1} &= i_{i2}^2 R_{i2} = \frac{i_{o1}^2 |Z_{o1}|^2}{|Z_{o1} + Z_{i2}|^2} R_{i2} \\ &= \frac{|G_{m1}Z_{o1}|^2 v_s^2}{(R_{o1} + R_{i2})^2 + (X_{o1} + X_{i2})^2} R_{i2} \end{aligned} \tag{6.66}$$

The maximum value of P_{o1} is called the available output power P_{ao1} from the first stage. It is solved for by setting $\partial P_{o1}/\partial R_{i2} = 0$ and $\partial P_{o1}/\partial X_{i2} = 0$ and solving for R_{i2} and X_{i2}. The solutions are $R_{i2} = R_{o1}$ and $X_{i2} = -X_{o1}$, i.e. $Z_{i2} = Z_{o1}^*$. It follows that P_{ao1} is given by

$$P_{ao1} = \frac{|G_{m1}Z_{o1}|^2 v_s^2}{4R_{o1}} \tag{6.67}$$

The available power gain G_{a1} of the first stage is given by

$$G_{a1} = \frac{P_{ao1}}{P_{ai1}} = \frac{|G_{m1}Z_{o1}|^2 R_s}{R_{o1}} \tag{6.68}$$

Similarly, it can be shown that the available power gain G_{aj} of the jth stage is given by

$$G_{aj} = \frac{P_{aoj}}{P_{aij}} = \frac{|G_{mj}Z_{oj}|^2 R_{o(j-1)}}{R_{oj}} \tag{6.69}$$

With these definitions, it follows that Eq. (6.63) for F can be written

$$F = F_1 + \frac{F_2 - 1}{G_{a1}} + \cdots + \frac{F_N - 1}{G_{a1}G_{a2}\cdots G_{a(N-1)}} \tag{6.70}$$

This is the desired result.

If G_{a1} can be made large enough, the above equation implies that $F \simeq F_1$. However, increasing G_{a1} may not make $(F_2 - 1)/G_{a1}$ approach zero. For example, consider the case where $Z_{o1} = R_{o1} + j0$. In this case, G_{a1} is given by

$$G_{a1} = |G_{m1}|^2 R_s R_{o1} \tag{6.71}$$

If R_{o1} is increased, G_{a1} can be made arbitrarily large. The contribution to F by the second-stage noise is given by

$$\frac{F_2 - 1}{G_{a1}} = \frac{1}{4kT_0 R_s \Delta f G_{m1}^2} \left[\frac{v_{n2}^2}{R_{o1}^2} + \frac{2v_n i_n \operatorname{Re}(\gamma)}{R_{o1}} + i_{n2}^2 \right] \tag{6.72}$$

This cannot be made arbitrarily small by making R_{o1} arbitrarily large unless i_{n2}^2 is negligible.

Let T_n be the noise temperature of the overall amplifier and T_{nj} the noise temperature of the jth stage. Eq. (6.58) can be used to express the noise factors in Eq. (6.70) in terms of the noise temperatures. It follows that the noise temperature of the multistage amplifier is given by

$$T_n = T_{n1} + \frac{T_{n2}}{G_{a1}} + \cdots + \frac{T_{nN}}{G_{a1}G_{a2}\cdots G_{a(N-1)}} \tag{6.73}$$

Example 7 *Use Eqs. (6.70) and (6.58) to calculate the noise factor and noise temperature of the two-stage amplifier in Example 3 of Chapter 4 for which* $v_n = 5\,\mathrm{nV}$, $i_{n1} = 2\,\mathrm{pA}$, $\gamma = 0$, $\Delta f, = 1\,\mathrm{Hz}$, $R_s = 1\,\mathrm{k\Omega}$, $Z_{o1} = R_{o1} = 20\,\mathrm{k\Omega}$, $G_{m1} = 35^{-1}\,\mathrm{S}$, *and* $G_{m2} = 225^{-1}\,\mathrm{S}$.

Solution.

$$F_1 = \frac{v_{ni1}^2}{4kTR_s\Delta f} = 1 + \frac{v_n^2 + i_n^2 R_s^2}{4kTR_s\Delta f} = 2.81$$

$$F_2 = \frac{v_{ni2}^2}{4kTR_{o1}\Delta f} = 1 + \frac{v_n^2 + i_n^2 R_s^2}{4kTR_{o1}\Delta f} = 6.08$$

$$F = F_1 + \frac{(F_2 - 1)\,R_{o1}}{|G_{m1}Z_{o1}|^2\,R_s} = 2.81$$

$$T_n = (F-1)\,T_0 = 526\,\mathrm{K}$$

It follows that the noise factor is determined by the first stage.

6.6 Effect of a Matching Network on Noise

6.6.1 Thévenin Source

Figure 6.7 shows a lossless matching network between the input to an amplifier having an input impedance $Z_i = R_i + jX_i$ and a signal source having an output impedance $Z_s = R_s + jX_s$. Denote the input impedance of the matching network by $Z_{im} = R_{im} + jX_{im}$ and its output impedance by $Z_{om} = R_{om} + jX_{om}$. The total power delivered to the matching network by the source is

$$P_{im} = \left|\frac{V_s + V_{ts}}{Z_s + Z_{im}}\right|^2 \mathrm{Re}\,(Z_{im}) = \frac{v_s^2 + 4kTR_s\Delta f}{|Z_s + Z_{im}|^2} R_{im} \tag{6.74}$$

Let V_{is} and V_{its}, respectively, be the voltages at the amplifier input due to V_s and V_{ts}. The output power from the matching network is

$$P_{om} = \left|\frac{V_{is} + V_{its}}{Z_i}\right|^2 \mathrm{Re}\,(Z_i) = \frac{v_{is}^2 + v_{its}^2}{|Z_i|^2} R_i \tag{6.75}$$

Because the matching network is lossless, we have $P_{im} = P_{om}$. This leads to the relation

$$v_{is}^2 + v_{its}^2 = \frac{R_{im}}{R_i}\left|\frac{Z_i}{Z_s + Z_{im}}\right|^2 \left(v_s^2 + 4kTR_s\Delta f\right) \tag{6.76}$$

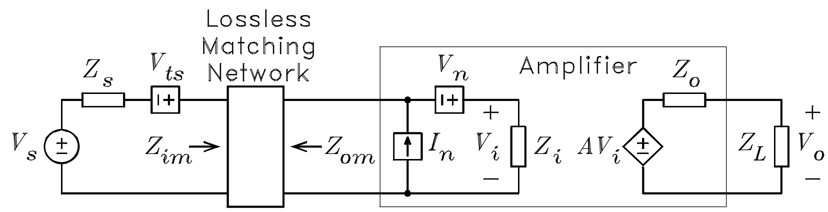

Figure 6.7: Amplifier with a Thévenin source and an input matching network.

Equation (6.76) can be rewritten

$$\begin{aligned} v_{is}^2 + v_{its}^2 &= \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \left[\frac{R_{im}}{R_i}\left|\frac{Z_i + Z_{om}}{Z_s + Z_{im}}\right|^2 \left(v_s^2 + 4kTR_s\Delta f\right)\right] \\ &= \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \left[v_{is(oc)}^2 + v_{its(oc)}^2\right] \end{aligned} \tag{6.77}$$

where $v_{is(oc)}^2$ and $v_{its(oc)}^2$, respectively, are the open-circuit values v_{is}^2 and v_{its}^2. The latter is given by

$$v_{its(oc)}^2 = \frac{R_{im}}{R_i}\left|\frac{Z_i + Z_{om}}{Z_s + Z_{im}}\right|^2 4kTR_s\Delta f \tag{6.78}$$

This equation must be of the form $v_{its(oc)}^2 = 4kTR_{om}\Delta f$, where $R_{om} = \mathrm{Re}\,(Z_{om})$. It follows that R_{om} is given by

$$R_{om} = \frac{v_{its(oc)}^2}{4kT\Delta f} = \frac{R_{im}}{R_i}\left|\frac{Z_i + Z_{om}}{Z_s + Z_{im}}\right|^2 R_s \tag{6.79}$$

When this equation is solved for R_{im}/R_i and the result used in Eq. (6.76), it follows that

$$v_{is}^2 + v_{its}^2 = \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \frac{R_{om}}{R_s}\left(v_s^2 + 4kTR_s\Delta f\right) \tag{6.80}$$

It might seem a contradiction that R_{om} in Eq. (6.79) is a function of Z_i. However, the dependence cancels because Z_{im} is also a function of Z_i. We conclude from Eq. (6.80) that the mean-square open-circuit output voltage from the matching network is given by

$$v_{is(oc)}^2 + v_{its(oc)}^2 = \frac{R_{om}}{R_s}\left(v_s^2 + 4kTR_s\Delta f\right) \tag{6.81}$$

To obtain the total mean-square open-circuit voltage at the input to the amplifier, the contributions of V_n and I_n must be added to Eq. (6.81). The result is

$$\begin{aligned} v_{i(oc)}^2 &= v_{is(oc)}^2 + v_{its(oc)}^2 + v_n^2 + 2v_ni_n\,\mathrm{Re}\,(\gamma Z_{om}^*) + i_n^2\,|Z_{om}|^2 \\ &= \frac{R_{om}}{R_s}\left(v_s^2 + 4kTR_s\Delta f\right) + v_n^2 \\ &\quad + 2v_ni_n\,\mathrm{Re}\,(\gamma Z_{om}^*) + i_n^2\,|Z_{om}|^2 \end{aligned} \tag{6.82}$$

It follows from this expression that the mean-square equivalent noise input voltage in series with V_s is

$$v_{ni}^2 = 4kTR_s\Delta f + \frac{R_s}{R_{om}}\left[v_n^2 + 2v_n i_n \operatorname{Re}(\gamma Z_{om}^*) + i_n^2 |Z_{om}|^2\right] \tag{6.83}$$

The noise factor is given by

$$F = \frac{v_{ni}^2}{4kTR_s\Delta f} = 1 + \frac{v_n^2 + 2v_n i_n \operatorname{Re}(\gamma Z_{om}^*) + i_n^2 |Z_{om}|^2}{4kTR_{om}\Delta f} \tag{6.84}$$

This is the same as the noise factor calculated at the output of the matching network. The basic reason that the noise factors at the source and at the output of the matching network are equal is because a lossless matching network cannot add noise. Thus it follows that the signal-to-noise ratio is also the same at the input to the matching network as it is at the input to the amplifier. However, these conclusions do not hold for a lossy matching network. Eq. (6.84) can be used to predict the noise factor for any arbitrary matching network. For example, Z_{om} might be chosen to be the optimum source impedance to minimize F. Alternately, it can be chosen for a conjugate impedance match to maximize the power gain. Such calculations are illustrated in Example 8.

For a conjugate match, the condition $Z_{om} = Z_i^*$ must hold. In this case, $R_{om} = R_i$ and the expression for v_{ni}^2 in Eq. (6.83) reduces to

$$v_{ni}^2 = 4kTR_s\Delta f + \frac{R_s}{R_i}\left[v_n^2 + 2v_n i_n \operatorname{Re}(\gamma Z_i) + i_n^2 |Z_i^*|^2\right] \tag{6.85}$$

The corresponding noise factor is

$$F = \frac{v_{ni}^2}{4kTR_s\Delta f} = 1 + \frac{v_n^2 + 2v_n i_n \operatorname{Re}(\gamma Z_i) + i_n^2 |Z_i^*|^2}{4kTR_i\Delta f} \tag{6.86}$$

Because of the dependence of v_{ni}^2 and F on Z_i, it is difficult to predict from these equations how changes in Z_i affect the noise. This is because V_n, I_n, and γ in the noise model are, in general, related to Z_i. For example, V_n, I_n, γ, and Z_i may all be functions of the bias current in the amplifier input stage. A change in the bias current to vary Z_i can cause a change in V_n, I_n, and γ. Thus the effects cannot be examined in detail unless the relations between the variables are known.

Example 8 *An amplifier is driven from a source with a resistive output impedance* $R_s = 50\,\Omega$*. At the operating frequency* $f = 10\,\text{MHz}$*, the amplifier has a resistive input impedance* $R_i = 25\,\Omega$ *and the noise parameters* $v_n/\sqrt{\Delta f} = 0.447\,\text{nV}/\sqrt{\text{Hz}}$*,* $i_n/\sqrt{\Delta f} = 31\,\text{pA}/\sqrt{\text{Hz}}$*, and* $\gamma = 0.12 - j0.44$*. (a) Calculate the noise figure with a conjugate impedance matching network between the source and the amplifier. (b) Calculate the noise figure if the matching network is designed so that the amplifier sees its optimum source impedance. (c) Calculate the decrease in power gain with the second matching network.*

Solution. (a) For a conjugate impedance match, the noise figure is

$$\begin{aligned} NF &= 10\log\left[1 + \frac{v_n^2 + 2v_n i_n \operatorname{Re}(\gamma Z_i) + i_n^2 |Z_i^*|^2}{4kTR_i\Delta f}\right] \\ &= 5.06\,\text{dB} \end{aligned}$$

(b) The optimum source impedance is given by Eq. (6.13). It is

$$Z_{opt} = \left[\sqrt{1-\gamma_i^2} - j\gamma_i\right]\frac{v_n}{i_n} = 13 + j6.35\,\Omega$$

Thus the minimum noise figure is

$$\begin{aligned} NF_{min} &= 10\log\left[1 + \frac{v_n^2 + 2v_n i_n \operatorname{Re}\left(\gamma Z_{opt}^*\right) + i_n^2\left|Z_{opt}\right|^2}{4kT\operatorname{Re}\left(Z_{opt}\right)\Delta f}\right] \\ &= 4.41\,\text{dB} \end{aligned}$$

This is 0.648 dB lower than for part (a).
(c) By Eq. (6.81), $v_{is(oc)}^2 = (R_{om}/R_s)\,v_s^2$ for part (a) is

$$v_{is(oc)}^2 = \frac{25}{50}v_s^2 = 0.5v_s^2$$

For part (b), it is

$$v_{is(oc)}^2 = \frac{13}{50}v_s^2 = 0.26v_s^2$$

The signal power delivered to the amplifier input is

$$P_i = i_i^2 R_i = \frac{v_{is(oc)}^2}{\left|Z_{om}+Z_i\right|^2}R_i$$

For part (a), we have

$$P_{i(a)} = \frac{0.5v_s^2}{\left|50+25\right|^2}25 = 0.005v_s^2$$

For part (b)

$$P_{i(b)} = \frac{0.26v_s^2}{\left|13+j6.35+25\right|^2}25 = 0.00439v_s^2$$

It follows that the amplifier power gain drops by the factor 0.00439/0.005 with the optimum source impedance. This is a decrease of 12.2% or 0.567 dB.

6.6.2 Norton Source

Figure 6.87 shows an amplifier with a Norton source at its input. The solutions for the mean-squared noise current in parallel with I_s and the noise factor follow the derivations of Eqs. (6.85) and (6.86) for the amplifier with a Thévenin source. The mean-square input noise current is given by

$$i_{ni}^2 = 4kTG_s\Delta f + \frac{G_s}{G_{om}}\left[v_n^2\left|Y_{om}\right|^2 + 2v_n i_n\operatorname{Re}\left(\gamma Y_{om}\right) + i_n^2\right] \tag{6.87}$$

where $G_s = \operatorname{Re}(Y_s)$ and $Y_{om} = G_{om} + jB_{om}$. The noise factor is given by

$$F = \frac{i_{ni}^2}{4kTG_s\Delta f} = 1 + \frac{v_n^2\left|Y_{om}\right|^2 + 2v_n i_n\operatorname{Re}\left(\gamma Y_{om}\right) + i_n^2}{4kTG_{om}\Delta f} \tag{6.88}$$

For a conjugate match, $Y_{om} = Y_i^*$. In this case i_{ni}^2 and F are given by

$$i_{ni}^2 = 4kTG_s\Delta f + \frac{G_s}{G_i}\left[v_n^2 \left|Y_i^*\right|^2 + 2v_n i_n \operatorname{Re}\left(\gamma Y_i^*\right) + i_n^2\right] \tag{6.89}$$

$$F = \frac{i_{ni}^2}{4kTG_s\Delta f} = 1 + \frac{v_n^2 \left|Y_i^*\right|^2 + 2v_n i_n \operatorname{Re}\left(\gamma Y_i^*\right) + i_n^2}{4kTG_i\Delta f} \tag{6.90}$$

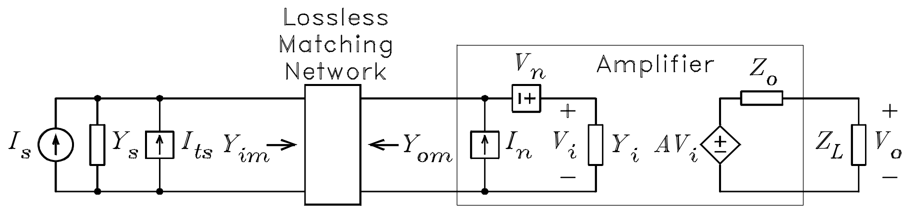

Figure 6.8: Amplifier with a Norton source and an input matching network.

6.7 Noise Circles

In rf design, the contours of constant F on the Smith chart for the reflection coefficient seen looking out of an amplifier input are important. These contours are circles. Thus they are called noise circles. These circles are developed below for both the impedance and admittance Smith charts.

6.7.1 Thévenin Source

Fig. 6.9 shows an amplifier with a Thévenin source and a lossless input matching network. Let the impedance seen looking into the output of the matching network be $Z_{om} = R_{om} + jX_{om}$. Imagine a zero length transmission line of characteristic impedance Z_c connected between the matching network and the amplifier input. Let Γ_{om} be the reflection coefficient seen looking into the output of the matching network. It is given by

$$\Gamma_{om} = \frac{Z_{om} - Z_c}{Z_{om} + Z_c} \tag{6.91}$$

By Eq. (6.27), the noise factor is given by

$$F = F_{min} + \frac{4G_n Z_c \left|\Gamma_{om} - \Gamma_{opt}\right|^2}{\left|1 - \Gamma_{opt}\right|^2 \left(1 - \left|\Gamma_{om}\right|^2\right)} \tag{6.92}$$

This equation can be rearranged into the form

$$\frac{\left|\Gamma_{om} - \Gamma_{opt}\right|^2}{1 - \left|\Gamma_{om}\right|^2} = z \tag{6.93}$$

where z is given by

$$z = \frac{\left(F - F_{min}\right)\left|1 - \Gamma_{opt}\right|^2}{4G_n Z_c} \tag{6.94}$$

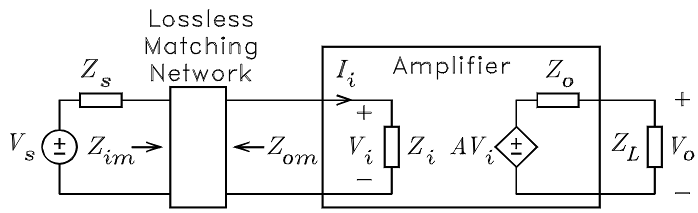

Figure 6.9: Amplifier with a Thévenin source and a lossless input matching network.

Let $\Gamma_{om} = p + jq$ and $\Gamma_{opt} = p_{opt} + jq_{opt}$. With these definitions, Eq. (6.93) can be written

$$p^2(1+z) + 2p_{opt}p + q^2(1+z) + 2q_{opt}q = z - |\Gamma_{opt}|^2 \tag{6.95}$$

After dividing both sides of this equation by $(1+z)$ and completing the squares, the equation can be reduced to

$$\left(p - \frac{p_{opt}}{1+z}\right)^2 + \left(q - \frac{q_{opt}}{1+z}\right)^2 = \frac{z}{1+z}\left(1 - \frac{|\Gamma_{opt}|^2}{1+z}\right) \tag{6.96}$$

or equivalently

$$\left|\Gamma_{om} - \frac{\Gamma_{opt}}{1+z}\right|^2 = \frac{z}{1+z}\left(1 - \frac{|\Gamma_{opt}|^2}{1+z}\right) \tag{6.97}$$

This equation represents a circle on the Smith chart with center at the point

$$\Gamma = a + jb = \frac{\Gamma_{opt}}{1+z} \tag{6.98}$$

and a radius given by

$$c = \left[\frac{z}{1+z}\left(1 - \frac{|\Gamma_{opt}|^2}{1+z}\right)\right]^{1/2} \tag{6.99}$$

For z a constant, Eq. (6.97) represents a circle of radius c that is centered at the point $\Gamma = \Gamma_{opt}/(1+z)$ on the Smith chart. On this circle, the noise factor is constant and is given by

$$F = F_{min} + \frac{4zG_nZ_c}{|1-\Gamma_{opt}|^2} = F_{min} + \frac{G_n}{R_{om}}|Z_{om} - Z_{opt}|^2 \tag{6.100}$$

Because F is constant for z a constant, it follows that the contours of constant F on the Smith chart are the circles defined by Eq. (6.97). For $Z_{om} = Z_{opt}$, the circles degenerate into a point located at $\Gamma = \Gamma_{opt}$. Because $b/a = \text{Im}(\Gamma_{opt}) / \text{Re}(\Gamma_{opt})$ is independent of Z_{om}, it follows that the centers of the noise circles lie on a straight line passing through the origin and the point Γ_{opt} on the Smith chart.

Figure 6.10(a) shows an example impedance Smith chart with the point $\Gamma_{opt} = 0.5\angle -135^\circ$ and three surrounding noise circles labeled Γ_1, Γ_2, and Γ_3, corresponding to $z = 0.3$, $z = 0.5$, and $z = 0.7$, respectively.

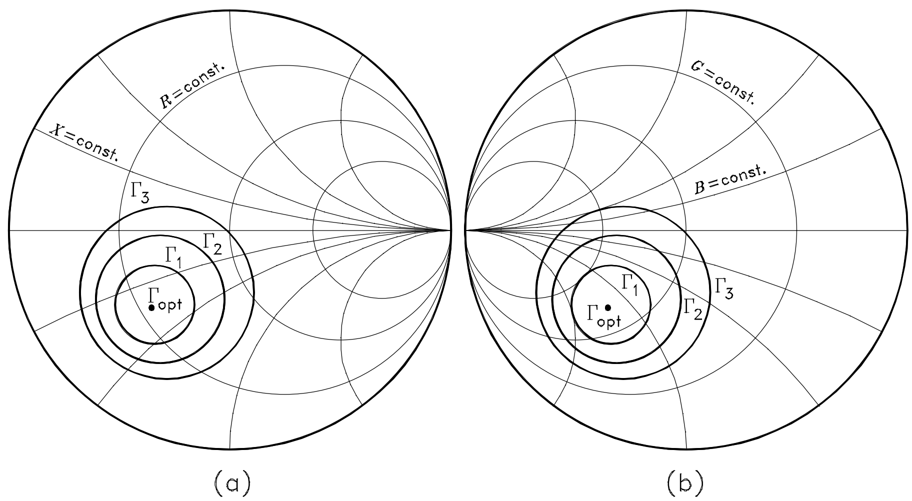

Figure 6.10: Smith charts showing the optimum reflection coefficient Γ_0 and three noise circles. (a) Impedance chart. (b) Admittance chart.

Example 9 *An amplifier is driven from a Thévenin source with an output impedance* $Z_s = R_s = 60\,\Omega$*. The amplifier has an input spot noise current* $i_n/\sqrt{\Delta f} = 20\,\text{pA}/\sqrt{\text{Hz}}$*. The optimum noise figure is* $NF_{min} = 1.5\,\text{dB}$ *when the source impedance is* $Z_{opt} = 30 - j20\,\Omega$*. (a) If a zero-length transmission line having a characteristic impedance* $Z_c = 50\,\Omega$ *is connected between the source and the amplifier, calculate the reflection coefficient* Γ_s *seen looking out of the amplifier input. (b) Calculate the optimum reflection coefficient* Γ_{opt}*. (c) Use Eq. (6.100) to calculate the noise figure when the amplifier is driven from the source. (d) Calculate the center coordinates and the radius of the noise circle that* Γ_{om} *lies on. (e) Calculate the equivalent spot noise input voltage* $v_{ni}/\sqrt{\Delta f}$*.*

Solution. (a) Looking out of the amplifier input, the reflection coefficient is

$$\Gamma_s = \frac{Z_s - Z_c}{Z_s + Z_c} = 0.091$$

(b) For $Z_s = Z_{opt}$, the optimum reflection coefficient is

$$\Gamma_{opt} = \frac{Z_{opt} - Z_c}{Z_{opt} + Z_c} = -0.176 - j0.294$$

(c)

$$G_n = \frac{i_n^2}{4kT_0\Delta f} = 0.025\,\text{S}$$

$$F_{min} = 10^{NF_{min}/10} = 1.413$$

$$F = F_{min} + \frac{G_n}{R_s}\left|R_s - Z_{opt}\right|^2 = 1.954$$

$$NF = 10\log(F) = 2.91\,\text{dB}$$

(d)

$$z = \frac{(F - F_{min})\left|1 - \Gamma_{opt}\right|^2}{4G_n Z_c} = 0.159$$

$$a = \frac{\mathrm{Re}\left(\Gamma_{opt}\right)}{1 + z} = -0.152$$

$$b = \frac{\mathrm{Im}\left(\Gamma_{opt}\right)}{1 + z} = -0.254$$

$$c = \left[\frac{z}{1 + z}\left(1 - \frac{\left|\Gamma_{opt}\right|^2}{1 + z}\right)\right]^{1/2} = 0.351$$

(e)

$$\frac{v_{ni}}{\sqrt{\Delta f}} = \sqrt{F \times 4kT_0 R_s} = 1.37\,\mathrm{nV}/\sqrt{\mathrm{Hz}}$$

6.7.2 Norton Source

Fig. 6.11 shows an amplifier with a Norton source and a lossless input matching network. Let the admittance seen looking into the output of the matching network be $Y_{om} = G_{om} + jB_{om}$. Imagine a zero length transmission line of characteristic admittance Y_c connected between the matching network and the amplifier input. Let Γ_{om} be the reflection coefficient seen looking into the output of the matching network. It is given by

$$\Gamma_{om} = \frac{Y_c - Y_{om}}{Y_c + Y_{om}} \tag{6.101}$$

By Eq. (6.44), the noise factor is given by

$$F = F_{min} + \frac{4R_n Y_c \left|\Gamma_{om} - \Gamma_{opt}\right|^2}{\left|1 + \Gamma_{opt}\right|^2 \left(1 - \left|\Gamma_{om}\right|^2\right)} \tag{6.102}$$

This equation can be rearranged into the form

$$\frac{\left|\Gamma_{om} - \Gamma_{opt}\right|^2}{1 - \left|\Gamma_{om}\right|^2} = y \tag{6.103}$$

where y is given by

$$y = \frac{(F - F_{min})\left|1 + \Gamma_{opt}\right|^2}{4R_n Y_c} \tag{6.104}$$

Let $\Gamma_{om} = p + jq$ and $\Gamma_{opt} = p_{opt} + jq_{opt}$. With these definitions, Eq. (6.103) can be written

$$p^2(1 + y) + 2p_{opt}p + q^2(1 + y) + 2q_{opt}q = y - \left|\Gamma_{opt}\right|^2 \tag{6.105}$$

After dividing both sides of this equation by $(1 + y)$ and completing the squares, the equation can be reduced to

$$\left(p - \frac{p_{opt}}{1 + y}\right)^2 + \left(q - \frac{q_{opt}}{1 + y}\right)^2 = \frac{y}{1 + y}\left(1 - \frac{\left|\Gamma_{opt}\right|^2}{1 + y}\right) \tag{6.106}$$

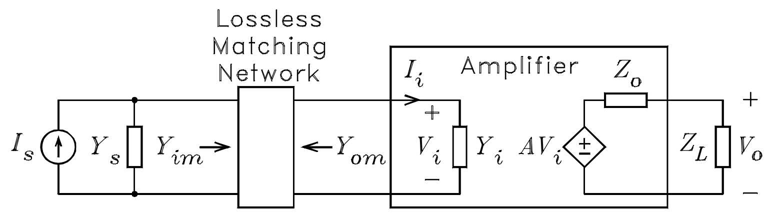

Figure 6.11: Amplifier with a Norton source and a lossless input matching network.

or equivalently

$$\left|\Gamma_{om} - \frac{\Gamma_{opt}}{1+y}\right|^2 = \frac{y}{1+y}\left(1 - \frac{|\Gamma_{opt}|^2}{1+y}\right) \tag{6.107}$$

This equation represents a circle on the Smith chart with a center at the point

$$\Gamma = a + jb = \frac{\Gamma_{opt}}{1+y} \tag{6.108}$$

and a radius given by

$$c = \left[\frac{y}{1+y}\left(1 - \frac{|\Gamma_{opt}|^2}{1+y}\right)\right]^{1/2} \tag{6.109}$$

For y a constant, Eq. (6.107) represents a circle of radius c that is centered at the point $\Gamma = \Gamma_{opt}/(1+y)$ on the Smith chart. On this circle, the noise factor is constant and is given by

$$F = F_{min} + \frac{4yR_nY_c}{|1+\Gamma_{opt}|^2} = F_{min} + \frac{R_n}{G_{om}}|Y_{om} - Y_{opt}|^2 \tag{6.110}$$

Because F is constant for y a constant, it follows that the contours of constant F on the Smith chart are the circles defined by Eq. (6.107). For $Y_{om} = Y_{opt}$, the circles degenerate into a point located at $\Gamma = \Gamma_{opt}$. Because $b/a = \mathrm{Im}(\Gamma_{opt})/\mathrm{Re}(\Gamma_{opt})$ is independent of Y_{om}, it follows that the centers of the noise circles lie on a straight line passing through the origin and the point Γ_{opt} on the Smith chart.

Figure 6.10(b) shows an example admittance Smith chart with the point $\Gamma_{opt} = 0.5\angle - 135^\circ$ and three surrounding noise circles labeled Γ_1, Γ_2, and Γ_3, corresponding to $y = 0.568$, $y = 0.983$, and $y = 1.33$, respectively. These values are chosen to give the same radii of the corresponding circles on the two charts in the figure. It follows from Eqs. (6.100) and (6.110), that the value of F on corresponding circles is not the same unless the following relation holds:

$$\frac{G_n}{R_{om}}|Z_{om} - Z_{opt}|^2 = \frac{R_n}{G_{om}}|Y_{om} - Y_{opt}|^2 \tag{6.111}$$

6.8 Gain Circles

Compromises are often made in rf amplifier design between lowest noise and highest gain. In Sec. 6.7, the contours on the Smith chart of constant noise factor are derived. In this section, the contours of constant gain are derived. Like the noise factor contours, the constant gain contours are circles. Thus they are called gain circles. Given the Smith chart with both the noise circles and the gain circles plotted for a particular amplifier, the effect of the input matching network on both noise and gain can be easily visualized.

6.8.1 Thévenin Source

Consider the amplifier model of Fig. 6.9 where the source is represented by a Thévenin equivalent. By Eq. (6.80), the mean-square signal voltage across Z_i is given by

$$\overline{v_i^2} = \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \frac{R_{om}}{R_s}\overline{v_s^2} \tag{6.112}$$

where $R_{om} = \mathrm{Re}\,(Z_{om})$ and $R_s = \mathrm{Re}\,(Z_s)$. The power delivered to Z_i is given by

$$P_i = \overline{i_i^2}\,\mathrm{Re}\,(Z_i) = \frac{\overline{v_i^2}}{|Z_i|^2}R_i = \frac{\overline{v_s^2}}{|Z_i + Z_{om}|^2}\frac{R_i R_{om}}{R_s} \tag{6.113}$$

The maximum value of P_i occurs when $Z_{om} = Z_i^*$ and is given by

$$P_{i(max)} = \frac{\overline{v_s^2}}{4R_s} \tag{6.114}$$

Thus the relative efficiency η of the input matching network can be written

$$\eta = \frac{P_i}{P_{i(max)}} = \frac{4R_i R_{om}}{|Z_i + Z_{om}|^2} \tag{6.115}$$

where $0 \leq \eta \leq 1$.

Let us next express Z_i, R_i, Z_{om}, and R_{om} in Eq. (6.115) as functions of reflection coefficients. Imagine a zero length transmission line of characteristic impedance Z_c connected between the matching network and the amplifier input. Let Γ_{om} be the reflection coefficient looking into the output of the matching network and let Γ_i be the reflection coefficient looking into the amplifier input. We can write

$$Z_i = Z_c\frac{1+\Gamma_i}{1-\Gamma_i} \tag{6.116}$$

$$R_i = \frac{Z_c}{2}\left(\frac{1+\Gamma_i}{1-\Gamma_i} + \frac{1+\Gamma_i^*}{1-\Gamma_i^*}\right) = Z_c\frac{1-|\Gamma_i|^2}{|1-\Gamma_i|^2} \tag{6.117}$$

$$Z_{om} = Z_c\frac{1+\Gamma_{om}}{1-\Gamma_{om}} \tag{6.118}$$

$$R_{om} = \frac{Z_c}{2}\left(\frac{1+\Gamma_{om}}{1-\Gamma_{om}} + \frac{1+\Gamma_{om}^*}{1-\Gamma_{om}^*}\right) = Z_c\frac{1-|\Gamma_{om}|^2}{|1-\Gamma_{om}|^2} \tag{6.119}$$

$$Z_i + Z_{om} = Z_c\left(\frac{1+\Gamma_i}{1-\Gamma_i} + \frac{1+\Gamma_{om}}{1-\Gamma_{om}}\right) = 2Z_c\frac{1-\Gamma_i\Gamma_{om}}{(1-\Gamma_i)(1-\Gamma_{om})} \tag{6.120}$$

With the use of these relations, Eq. (6.115) can be written

$$\eta = \frac{\left(1-|\Gamma_i|^2\right)\left(1-|\Gamma_{om}|^2\right)}{|1-\Gamma_i\Gamma_{om}|^2} \tag{6.121}$$

With the substitutions $\Gamma_{om} = p + jq$ and $\Gamma_i = p_i + jq_i$ the above equation becomes

$$p^2 - 2\frac{\eta p_i}{1-(1-\eta)\left|\Gamma_i\right|^2}p + q^2 + 2\frac{\eta q_i}{1-(1-\eta)\left|\Gamma_i\right|^2}q = \frac{1-\eta-\left|\Gamma_i\right|^2}{1-(1-\eta)\left|\Gamma_i\right|^2} \tag{6.122}$$

After completing the squares, this equation can be reduced to

$$\left|\Gamma_{om} - \frac{\eta\Gamma_i^*}{1-(1-\eta)\left|\Gamma_i\right|^2}\right|^2 = (1-\eta)\left[\frac{1-\left|\Gamma_i\right|^2}{1-(1-\eta)\left|\Gamma_i\right|^2}\right]^2 \tag{6.123}$$

This represents the equation of a circle on the Smith chart with a center at the point

$$\Gamma = a + jb = \frac{\eta\Gamma_i^*}{1-(1-\eta)\left|\Gamma_i\right|^2} \tag{6.124}$$

and a radius given by

$$c = \frac{1-\left|\Gamma_i\right|^2}{1-(1-\eta)\left|\Gamma_i\right|^2}\sqrt{1-\eta} \tag{6.125}$$

Because $b/a = \operatorname{Im}\left(\Gamma_i^*\right)/\operatorname{Re}\left(\Gamma_i^*\right)$ is independent of η, it follows that the centers of the gain circles lie on a straight line passing through the origin and the point Γ_i^* on the Smith chart.

Figure 6.12(a) shows an example impedance Smith chart with the point $\Gamma_i^* = 0.3\angle 45^\circ$ labeled η_0 and three surrounding gain circles labeled η_1, η_2, and η_3, corresponding to gains lower than η_0 by 1 dB, 2 dB, and 3 dB, respectively.

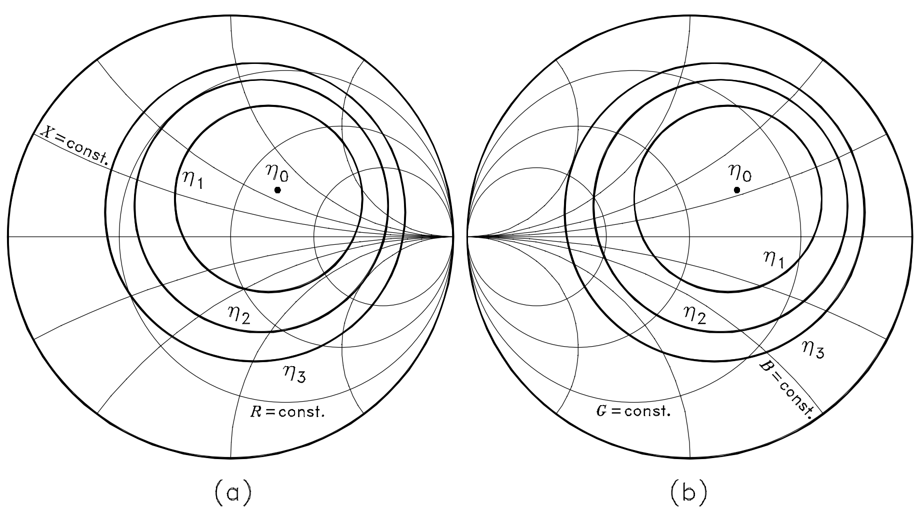

Figure 6.12: (a) Impedance Smith chart showing example gain circles corresponding to 0, −1, −2, and −3 dB. (b) Corresponding admittance Smith chart.

Example 10 *The amplifier of Example 1 in Chapter 5 is driven from a Thévenin source with the output impedance* $Z_s = R_s = 50\,\Omega$*. The amplifier has the input impedance* $Z_i = R_i + jX_i = 31.6 - j127\,\Omega$ *and a gain of* 39.8 dB*. (a) Calculate the value of* η *if no matching network is used at* *the amplifier input. (b) Calculate a, b, and c for the gain circle on which η lies. (c) Calculate the new overall gain of the amplifier if a conjugate matching network is added between the source and the amplifier. Assume zero-length transmission lines with the characteristic impedance $Z_c = 50\,\Omega$ for the Smith chart calculations.*

Solution. (a) Because there is no input matching network, $Z_{om} = Z_s = R_s = 50\,\Omega$.

$$\eta = \frac{4\,\mathrm{Re}\,(Z_i)\,\mathrm{Re}\,(Z_{om})}{|Z_i + Z_{om}|^2} = 0.277 \text{ or } -5.57\,\mathrm{dB}$$

(b)

$$\Gamma_i = \frac{Z_i - Z_c}{Z_i + Z_c} = 0.642 - j0.557$$

$$a = \frac{\eta\,\mathrm{Re}\,(\Gamma_i^*)}{1 - (1-\eta)\,|\Gamma_i|^2} = 0.373$$

$$b = \frac{\eta\,\mathrm{Im}\,(\Gamma_i^*)}{1 - (1-\eta)\,|\Gamma_i|^2} = 0.324$$

$$c = \frac{1 - |\Gamma_i|^2}{1 - (1-\eta)\,|\Gamma_i|^2}\sqrt{1-\eta} = 0.493$$

(c) The addition of a conjugate matching network at the input would increase the dB gain by $10\log(1/\eta)$ to $39.8 + 5.57 = 45.4\,\mathrm{dB}$.

6.8.2 Norton Source

Consider the amplifier model of Fig. 6.11, where the source is represented by a Norton equivalent. Let $Y_s = G_s + jB_s$, $Y_{om} = G_{om} + jB_{om}$, and $Y_i = G_i + jB_i$. Eq. (6.112) can be transformed into the equivalent equation for the Norton source with the substitutions $Z_i = 1/Y_i$, $Z_{om} = 1/Y_{om}$, $R_{om} = G_{om}/\,|Y_{om}|^2$, $R_s = G_s/\,|Y_s|^2$, $v_i^2 = i_i^2/\,|Y_i|^2$, and $v_s^2 = i_s^2/\,|Y_s|^2$. After these substitutions are made, it follows from Eq. (6.112) that the mean-square signal current through Y_i is given by

$$i_i^2 = \left|\frac{Y_i}{Y_i + Y_{om}}\right|^2 \frac{G_{om}}{G_s} i_s^2 \tag{6.126}$$

The power delivered to Y_i is given by

$$P_i = i_i^2\,\mathrm{Re}\left(\frac{1}{Y_i}\right) = i_i^2 \frac{G_i}{|Y_i|^2} = \frac{i_s^2}{|Y_i + Y_{om}|^2}\,\frac{G_i G_{om}}{G_s} \tag{6.127}$$

The maximum value of P_i occurs when $Y_{om} = Y_i^*$ and is given by

$$P_{i(max)} = \frac{i_s^2}{4G_s} \tag{6.128}$$

Thus the relative efficiency η of the input matching network can be written

$$\eta = \frac{P_i}{P_{i(max)}} = \frac{4G_i G_{om}}{|Y_i + Y_{om}|^2} \tag{6.129}$$

With this value of η, the equations for the gain circles for the Norton source are the same as they are for the Thévenin source given by Eqs. (6.123) through (6.125). This follows because the gain circle equations involve only the power ratio η and the input reflection coefficient Γ_i. Fig. 6.12(b) shows the gain circles on the admittance Smith chart corresponding to those in Fig. 6.12(a) on the impedance Smith chart.

Example 11 *An amplifier designed to operate at the frequency* $f = 1.9\,\text{GHz}$ *from a Norton source with an output impedance* $Z_s = R_s = 50\,\Omega$ *has the specifications: optimum source reflection coefficient for minimum noise* $\Gamma_{opt} = 0.52\angle 68.8^\circ$*, minimum noise figure* $NF_{min} = 1.39\,\text{dB}$*, noise resistance* $R_n = 20.4\,\Omega$*, input reflection coefficient* $\Gamma_i = 0.68\angle -86^\circ$*. The reflection coefficients are measured with a test fixture having the characteristic impedance* $Z_c = 50\,\Omega$*. (a) A noise matching network is to be used between the source and the amplifier. On an admittance Smith chart, plot the point representing* Γ_{opt} *and the noise circle for which the noise is* 0.25 *dB higher than its minimum value. (b) A conjugate impedance matching network is to be used between the source and the amplifier. On the same chart, plot the point representing the reflection coefficient seen looking out of the amplifier input and the gain circle for which the gain is* 1 dB *lower than its maximum value.*

Solution. (a) The point representing Γ_{opt} is shown on the chart in Fig. 6.13. At this point, the noise factor is

$$F_{min} = 10^{NF_{min}/10} = 1.377$$

For 0.25 dB higher noise, the noise figure is $NF_1 = NF_{min} + 0.25 = 1.64\,\text{dB}$. The corresponding noise factor is

$$F_1 = 10^{NF_1/10} = 1.459$$

Eq. (6.104) can be used to calculate y to obtain

$$y = \frac{(F_1 - F_{min})\,|1 + \Gamma_{opt}|^2}{4R_n Y_c} = 0.082$$

where $Y_c = 1/Z_c = 0.02\,\text{S}$. The coordinates (a, b) of the center of the $-0.25\,\text{dB}$ noise circle and its radius c are calculated from Eqs. (6.108) and (6.109) as follows:

$$a = \frac{\text{Re}\,(\Gamma_{opt})}{1 + y} = 0.174$$

$$b = \frac{\text{Im}\,(\Gamma_{opt})}{1 + y} = 0.448$$

$$c = \left[\frac{y}{1 + y}\left(1 - \frac{|\Gamma_{opt}|^2}{1 + y}\right)\right]^{1/2} = 0.239$$

The noise circle is shown labeled Γ_1 in Fig. 6.13.

(b) For a conjugate impedance matching network, the reflection coefficient seen looking out of the amplifier input is $\Gamma_{om} = \Gamma_i^* = 0.68\angle +86^\circ = 0.047 + j0.678$. This reflection coefficient maximizes the normalized gain of the matching network and is labeled η_0 in Fig. 6.13. On the gain circle for which the normalized gain of the matching network is -1 dB, the value of η in Eq. (6.129) is

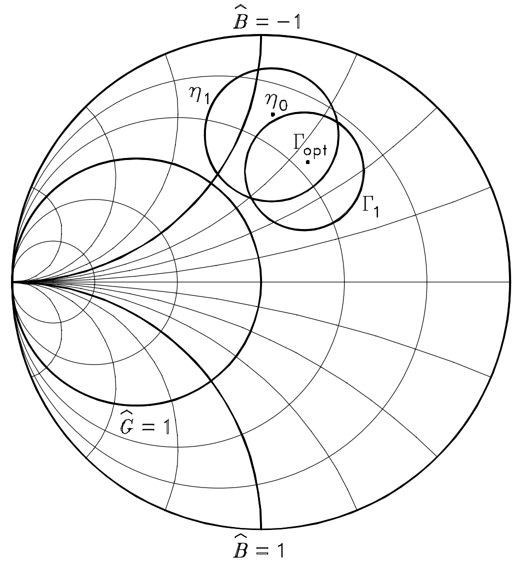

Figure 6.13: Admittance Smith chart for Example 11.

$\eta_1 = 10^{-1/10} = 0.794$. The coordinates (a, b) of the center of the -1 dB gain circle and its radius c are calculated from Eqs. (6.124) and (6.125) as follows:

$$a = \frac{\eta \operatorname{Re}(\Gamma_i^*)}{1 - (1-\eta)|\Gamma_i|^2} = 0.042$$

$$b = \frac{\eta \operatorname{Im}(\Gamma_i^*)}{1 - (1-\eta)|\Gamma_i|^2} = 0.595$$

$$c = \frac{1 - |\Gamma_i|^2}{1 - (1-\eta)|\Gamma_i|^2}\sqrt{1-\eta} = 0.269$$

The gain circle is shown labeled η_1 in Fig. 6.13. If the source admittance lies in the intersection of the noise circle and the gain circle, the noise factor is within 0.25 dB of its minimum value and the relative power gain of the input matching network is within 1 dB of its maximum value.

6.9 Measuring the Noise Factor

6.9.1 Method 1

This method is the most general one because it does not require knowledge of either the amplifier gain or its noise bandwidth. Consider the noise model of an amplifier given in Fig. 6.14. Consider the source to be a white noise source having the spectral density $S_v(f) = \overline{V_s V_s^*}/\Delta f = v_s^2/\Delta f$ and output resistance R_s. The total noise voltage at the output can be written

$$\begin{aligned} V_o &= A\frac{Z_L}{Z_o + Z_L}\left[(V_s + V_{ts} + V_n)\frac{Z_i}{R_s + Z_i} + I_n(R_s \| Z_i)\right] \\ &= \frac{AZ_L}{Z_o + Z_L}\,\frac{Z_i}{R_s + Z_i}(V_s + V_{ts} + V_n + I_n R_s) \end{aligned} \tag{6.130}$$

The mean-square value is given by

$$\begin{aligned} v_o^2 &= \left| \frac{AZ_L}{Z_o + Z_L} \frac{Z_i}{R_s + Z_i} \right|^2 \Big[S_v(f) B_n + 4kTR_s B_n + v_n^2 \\ &\quad +2 v_n i_n \operatorname{Re}(\gamma R_s) + i_n^2 |R_s|^2 \Big] \\ &= \left| \frac{AZ_L}{Z_o + Z_L} \frac{Z_i}{R_s + Z_i} \right|^2 \left[S_v(f) B_n + F \times 4kTR_s B_n \right] \end{aligned} \tag{6.131}$$

where B_n is the amplifier noise bandwidth and F is the noise factor given by Eq. (6.11).

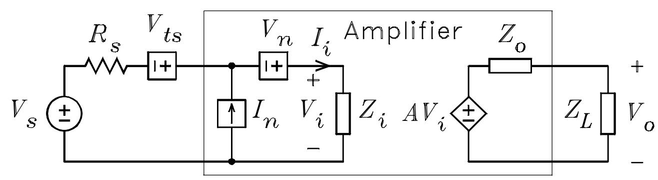

Figure 6.14: Amplifier driven by a white noise source.

Let v_{o1}^2 be the value of v_o^2 with the noise source at the input set to zero, i.e. $S_v(f) = 0$. Now, let $S_v(f)$ be increased until the rms output voltage increases by a factor r, i.e. $v_o = rv_{o1}$. It follows by taking the ratio of the two mean-square voltages that

$$r^2 = 1 + \frac{S_v(f) B_n}{F \times 4kTR_s B_n} = 1 + \frac{S_v(f)}{F \times 4kT_0 R_s} \tag{6.132}$$

The above equation can be solved for F to obtain

$$F = \frac{S_v(f)}{(r^2 - 1) \times 4kT_0 R_s} \tag{6.133}$$

In making measurements, a commonly used value for r is $r = \sqrt{2}$. In this case, the output noise voltage increases by 3 dB when the source is activated. Note that the expression for F is independent of B_n, A, and Z_i.

6.9.2 Method 2

This method replaces the white noise source in Fig. 6.14 with a sinusoidal source. The frequency should be chosen for maximum gain. The noise bandwidth B_n of the amplifier must be known. An often used alternative is to estimate B_n with the equation

$$B_n = \frac{\pi}{2} B_3 \tag{6.134}$$

where B_3 is the -3 dB bandwidth. This expression is exact if the amplifier has a first-order low-pass response or a second-order band-pass response.

The mean-square output voltage is given by

$$v_o^2 = \left|\frac{AZ_L}{Z_o+Z_L}\,\frac{Z_i}{R_s+Z_i}\right|^2 \Big[v_s^2 + 4kTR_sB_n + v_n^2 + 2v_n i_n \operatorname{Re}(\gamma R_s) + i_n^2 |R_s|^2 \Big] \tag{6.135}$$

where v_s^2 is the mean-square open-circuit source voltage. With $V_s = 0$, the mean-square noise output voltage is measured. Denote this by v_{o1}^2. Increase V_s until the rms output voltage increases by the factor r. It follows by taking the ratio of the two mean-square voltages that

$$\begin{aligned} r^2 &= 1 + \frac{v_s^2}{4kTR_sB_n + v_n^2 + 2v_n i_n \operatorname{Re}(\gamma R_s) + i_n^2 |R_s|^2} \\ &= 1 + \frac{v_s^2}{F \times 4kT_0B_nR_s} \end{aligned} \tag{6.136}$$

Solution for F yields

$$F = \frac{v_s^2}{(r^2-1) \times 4kT_0B_nR_s} \tag{6.137}$$

This method requires knowledge of the noise bandwidth of the amplifier.

6.9.3 Method 3

Unlike the above methods, this method requires knowledge of both B_n and the amplifier gain. The gain is measured with a sine-wave source having an open-circuit output voltage V_s and an output resistance R_s, where R_s is the value of the source resistance for which the noise factor is to be measured. With the source connected to the amplifier input, adjust the voltage to obtain a convenient voltage at the amplifier output. Denote this by V_{o1}. The frequency should be chosen for maximum gain. Next, disconnect the source from the amplifier input and measure its open-circuit output voltage. Denote this by V_{s1}. Let A_0 be the magnitude of the gain at the test frequency. With reference to the model in Fig. 6.14, it is given by

$$A_0 = \left|\frac{V_{o1}}{V_{s1}}\right| = \left|\frac{AZ_L}{Z_o+Z_L}\,\frac{Z_i}{R_s+Z_i}\right| \tag{6.138}$$

This is the gain including the loading effects at the input and at the output. The next step is to measure the noise bandwidth B_n. An often used alternative is to estimate B_n with Eq. (6.134).

The source is then replaced with a resistor of value R_s and the amplifier noise output voltage is measured. The mean-square value is given by

$$v_{no}^2 = \left|\frac{AZ_L}{Z_o+Z_L}\,\frac{Z_i}{R_s+Z_i}\right|^2 v_{ni}^2 = 4kT_0R_sB_nFA_0^2 \tag{6.139}$$

This equation can be solved for F to obtain

$$F = \frac{v_{no}^2}{4kT_0R_sB_nA_0^2} \tag{6.140}$$

6.10 Determination of Noise Parameters

Let the noise factor F be measured for N values of source admittance, where $N \geq 4$. Denote the noise factor values by F_i and the source admittance values by $Y_{si} = G_{si} + jB_{si}$, where $1 \leq i \leq N$. By Eq. (6.29), we can write

$$F_i = 1 + R_n \left(G_{si} + \frac{B_{si}^2}{G_{si}} \right) + 2R_n G_\gamma - 2R_n B_\gamma \frac{B_{si}}{G_{si}} + \frac{G_n}{G_{si}} \tag{6.141}$$

where $Y_\gamma = G_\gamma + jB_\gamma$ is the correlation admittance given by Eq. (6.10). The object is to use the measured values of F to determine the noise resistance R_n, the noise conductance G_n, and the correlation admittance Y_γ.

Define the mean-square error function

$$\begin{aligned} \epsilon^2 &= \sum_{i=1}^{N} \left[(F_i - 1) - R_n \left(G_{si} + \frac{B_{si}^2}{G_{si}} \right) - 2R_n G_\gamma \right. \\ &\quad \left. + 2R_n B_\gamma \frac{B_{si}}{G_{si}} - \frac{G_n}{G_{si}} \right]^2 \end{aligned} \tag{6.142}$$

The values of R_n, G_n, G_γ, and B_γ which minimize ϵ^2 represent a best estimate of the noise parameters. These values can be obtained by simultaneous solution of the set of equations $\partial\epsilon^2/\partial R_n = 0$, $\partial\epsilon^2/\partial G_n = 0$, $\partial\epsilon^2/\partial(R_n G_\gamma) = 0$, and $\partial\epsilon^2/\partial(R_n B_\gamma) = 0$, where $R_n G_\gamma$ and $R_n B_\gamma$ are considered independent variables. This procedure leads to the following solution:

$$\begin{bmatrix} R_n \\ 2R_n G_\gamma \\ -2R_n B_\gamma \\ G_n \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \sum \left(G_i + \frac{B_i^2}{G_i} \right) (F_i - 1) \\ \sum (F_i - 1) \\ \sum \frac{B_i}{G_i} (F_i - 1) \\ \sum \frac{1}{G_i} (F_i - 1) \end{bmatrix} \tag{6.143}$$

where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} \sum \left(G_{si} + \frac{B_{si}^2}{G_{si}} \right)^2 & \sum G_{si} + \frac{B_{si}^2}{G_{si}} & \sum B_i + \frac{B_{si}^3}{G_{si}^2} & \sum 1 + \frac{B_{si}^2}{G_{si}^2} \\ \sum G_{si} + \frac{B_{si}^2}{G_{si}} & N & \sum \frac{B_{si}}{G_{si}} & \sum \frac{1}{G_{si}} \\ \sum B_{si} + \frac{B_{si}^3}{G_{si}^2} & \sum \frac{B_{si}}{G_{si}} & \sum \frac{B_{si}^2}{G_{si}^2} & \sum \frac{B_{si}}{G_{si}^2} \\ \sum 1 + \frac{B_{si}^2}{G_{si}^2} & \sum \frac{1}{G_{si}} & \sum \frac{B_{si}}{G_{si}^2} & \sum \frac{1}{G_{si}^2} \end{bmatrix} \tag{6.144}$$

The matrix $\mathbf{A}$ is singular if the ratios $a_{m_1,n}/a_{m_2,n}$ are equal for all elements in any two rows. With 4 rows, there are 6 combinations of two rows. It follows that the matrix is singular if the values of G_i and B_i lie on one of the contours defined by

$$G^2 + B^2 = k_1^2 \tag{6.145}$$

$$(G - k_2)^2 + B^2 = k_2^2 \tag{6.146}$$

$$G + (B - k_3)^2 = k_3^2 \tag{6.147}$$

$$B = k_4 G \tag{6.148}$$

$$G = k_5 \tag{6.149}$$

$$B = k_6 \tag{6.150}$$

where k_1 through k_6 are constants. Fig. 6.15 shows example plots of these equations on the (G, B) plane. The k's are chosen so that the curves intersect at two common points. The curves are labeled a through f, corresponding, in order, to Eqs. (6.145) through (6.150). Two curves are labeled c, d, and f, corresponding to positive and negative values of k_3, k_4, and k_6.

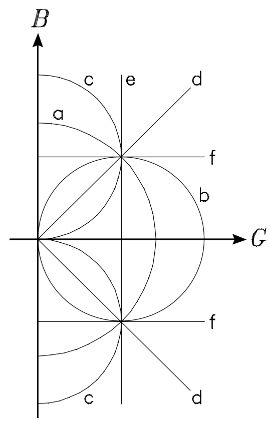

Figure 6.15: Example contours on which the matrix **A** is singular.

Given the solution for R_n, G_n, G_γ, and B_γ, the solutions for v_n^2, i_n^2, and γ are

$$v_n^2 = 4kT_0 R_n \Delta f \tag{6.151}$$

$$i_n^2 = 4kT_0 G_n \Delta f \tag{6.152}$$

$$\gamma = \frac{v_n}{i_n}\left(G_\gamma - jB_\gamma\right) \tag{6.153}$$

Because the quantity $(F - 1)$ is involved in the calculations, large percentage errors in $(F - 1)$ can be caused by small percentage errors in F when F has a value close to 1. Thus the solutions can be sensitive to experimental errors. Another problem lies in the choice of the values of G_i and B_i for which F is measured. If the values lie on or near one of the curves which makes the **A** matrix singular, the solutions can be unstable. To minimize this problem, the values of G_i and B_i should be chosen randomly.